

# Magnetic Strings in Dilaton Gravity

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First, I present two new classes of magnetic rotating solutions in four-dimensional Einstein-Maxwell-dilaton gravity with Liouville-type potential. The first class of solutions yields a 4-dimensional spacetime with a longitudinal magnetic field generated by a static or spinning magnetic string. I find that these solutions have no curvature singularity and no horizons, but have a conic geometry. In these spacetimes, when the rotation parameter does not vanish, there exists an electric field, and therefore the spinning string has a net electric charge which is proportional to the rotation parameter. The second class of solutions yields a spacetime with an angular magnetic field. These solutions have no curvature singularity, no horizon, and no conical singularity. The net electric charge of the strings in these spacetimes is proportional to their velocities. Second, I obtain the  $(n + 1)$ -dimensional rotating solutions in Einstein-dilaton gravity with Liouville-type potential. I argue that these solutions can present horizonless spacetimes with conic singularity, if one chooses the parameters of the solutions suitable. I also use the counterterm method and compute the conserved quantities of these spacetimes.

## I. INTRODUCTION

It seems likely, at least at sufficiently high energy scales, that gravity is not governed by the Einstein's action, and is modified by the superstring terms which are scalar tensor in nature. In the low energy limit of the string theory, one recovers Einstein gravity along with a scalar dilaton field which is non minimally coupled to the gravity [1]. Scalar-tensor theories are not new, and it was pioneered by Brans and Dicke [2], who sought to incorporate Mach's principle into gravity.

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In this paper I want to obtain new exact horizonless solutions of four-dimensional Einstein-Maxwell-dilaton gravity and higher dimensional Einstein-dilaton gravity. There are many papers which are dealing directly with the issue of spacetimes generated by string source that are horizonless and have non trivial external solutions. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions with vanishing cosmological constant have been considered in [3]. Similar static solutions in the context of cosmic string theory have been found in [4]. All of these solutions [3, 4] are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. An extension to include the electromagnetic field has also been done [5]. Asymptotically anti de Sitter (AdS) spacetimes generated by static and spinning magnetic sources in three and four dimensional Einstein-Maxwell gravity with negative cosmological constant have been investigated in [6]. The generalization of these asymptotically AdS magnetic rotating solutions of the Einstein-Maxwell equation to higher dimensions [7] and higher derivative gravity [8] have also been done. In the context of electromagnetic cosmic string, it was shown that there are cosmic strings, known as superconducting cosmic string, that behave as superconductors and have interesting interactions with astrophysical magnetic fields [9]. The properties of these superconducting cosmic strings have been investigated in [10]. Superconducting cosmic strings have also been studied in Brans-Dicke theory [11], and in dilaton gravity [12].

On the other side, some efforts have been done to construct exact solutions of Einstein-Maxwell-dilaton gravity. Exact charged dilaton black hole solutions in the absence of dilaton potential have been constructed by many authors [13, 14]. In the presence of Liouville potential, static charged black hole solutions have also been discovered with a positive constant curvature event horizons [15], and zero or negative constant curvature horizons [16]. These exact solutions are all static. Till now, charged rotating dilaton solutions for an arbitrary coupling constant has not been constructed in four or higher dimensions. Indeed, exact magnetic rotating solutions have been considered in three dimensions [17], while exact rotating black hole solutions in four dimensions have been obtained only for some limited values of the coupling constant [18]. For general dilaton coupling, the properties of rotating charged dilaton black holes only with infinitesimally small angular momentum [19] or small charge [20] have been investigated, while for arbitrary values of angular momentum and charge only a numerical investigation has been done [21]. My aim, here, is to construct exact rotating charged dilaton solutions for an arbitrary value of coupling constant.

The outline of the paper is as follows. Section II is devoted to a brief review of the field equations and general formalism of calculating the conserved quantities. In Sec III, I obtain the four-dimensional rotating dilaton strings which produce longitudinal and angular magnetic fields, and investigate their properties. In Sec. IV, I present the  $(n+1)$ -dimensional rotating dilaton solutions with conic singularity and compute their conserved charges. I finish the paper with some concluding remarks.

## II. FIELD EQUATIONS AND CONSERVED QUANTITIES

The action of dilaton Einstein-Maxwell gravity with one scalar field  $\Phi$  with Liouville-type potential in  $(n+1)$  dimensions is [15]

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-1} (\nabla\Phi)^2 - 2\Lambda e^{2\beta\Phi} - e^{-4\alpha\Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \Theta(\gamma), \quad (1)$$

where  $\mathcal{R}$  is the Ricci scalar,  $\alpha$  is a constant determining the strength of coupling of the scalar and electromagnetic fields,  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the electromagnetic tensor field and  $A_\mu$  is the vector potential. The last term in Eq. (1) is the Gibbons-Hawking boundary term. The manifold  $\mathcal{M}$  has metric  $g_{\mu\nu}$  and covariant derivative  $\nabla_\mu$ .  $\Theta$  is the trace of the extrinsic curvature  $\Theta^{\mu\nu}$  of any boundary(ies)  $\partial\mathcal{M}$  of the manifold  $\mathcal{M}$ , with induced metric(s)  $\gamma_{ij}$ . One may refer to  $\Lambda$  as the cosmological constant, since in the absence of the dilaton field ( $\Phi = 0$ ) the action (1) reduces to the action of Einstein-Maxwell gravity with cosmological constant.

The field equations in  $(n+1)$  dimensions are obtained by varying the action (1) with respect to the dynamical variables  $A_\mu$ ,  $g_{\mu\nu}$  and  $\Phi$ :

$$\partial_\mu [\sqrt{-g} e^{-4\alpha\Phi/(n-1)} F^{\mu\nu}] = 0, \quad (2)$$

$$\mathcal{R}_{\mu\nu} = \frac{2}{n-1} (2\partial_\mu \Phi \partial_\nu \Phi + \Lambda e^{2\beta\Phi} g_{\mu\nu}) + 2e^{-4\alpha\Phi/(n-1)} \left( F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2(n-1)} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \right) \quad (3)$$

$$\nabla^2 \Phi = \frac{n-1}{2} \beta \Lambda e^{2\beta\Phi} - \frac{\alpha}{2} e^{-4\alpha\Phi/(n-1)} F_{\rho\sigma} F^{\rho\sigma}. \quad (4)$$

The conserved mass and angular momentum of the solutions of the above field equations can be calculated through the use of the subtraction method of Brown and York [22]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. For asymptotically (A)dS solutions, the way that one deals with these divergences is

through the use of counterterm method inspired by (A)dS/CFT correspondence [23]. However, in the presence of a non-trivial dilaton field, the spacetime may not behave as either dS ( $\Lambda > 0$ ) or AdS ( $\Lambda < 0$ ). In fact, it has been shown that with the exception of a pure cosmological constant potential, where  $\beta = 0$ , no AdS or dS static spherically symmetric solution exist for Liouville-type potential [24]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence [25], there may be a suitable counterterm for the stress energy tensor which removes the divergences. In this paper, I deal with the spacetimes with zero curvature boundary [ $R_{abcd}(\gamma) = 0$ ], and therefore the counterterm for the stress energy tensor should be proportional to  $\gamma^{ab}$ . Thus, the finite stress-energy tensor in  $(n + 1)$  dimensions may be written as

$$T^{ab} = \frac{1}{8\pi} \left[ \Theta^{ab} - \Theta \gamma^{ab} + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right], \quad (5)$$

where  $l_{\text{eff}}$  is given by

$$l_{\text{eff}}^2 = \frac{(n-1)^3 \beta^2 - 4n(n-1)}{8\Lambda} e^{-2\beta\Phi}. \quad (6)$$

As  $\beta$  goes to zero, the effective  $l_{\text{eff}}^2$  of Eq. (6) reduces to  $l^2 = -n(n-1)/2\Lambda$  of the AdS spacetimes. The first two terms in Eq. (5) is the variation of the action (1) with respect to  $\gamma^{ab}$ , and the last term is the counterterm which removes the divergences. One may note that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where  $l$  is replaced by  $l_{\text{eff}}$ . To compute the conserved charges of the spacetime, one should choose a spacelike surface  $\mathcal{B}$  in  $\partial\mathcal{M}$  with metric  $\sigma_{ij}$ , and write the boundary metric in ADM form:

$$\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij} (d\varphi^i + V^i dt) (d\varphi^j + V^j dt),$$

where the coordinates  $\varphi^i$  are the angular variables parameterizing the hypersurface of constant  $r$  around the origin, and  $N$  and  $V^i$  are the lapse and shift functions respectively. When there is a Killing vector field  $\xi$  on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (5) can be written as

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \quad (7)$$

where  $\sigma$  is the determinant of the metric  $\sigma_{ij}$ , and  $\xi$  and  $n^a$  are the Killing vector field and the unit normal vector on the boundary  $\mathcal{B}$ . For boundaries with timelike ( $\xi = \partial/\partial t$ ), rotational

( $\varsigma = \partial/\partial\varphi$ ) and translational Killing vector fields ( $\zeta = \partial/\partial x$ ), one obtains the quasilocal mass, angular and linear momenta

$$M = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \quad (8)$$

$$J = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \zeta^b, \quad (9)$$

$$P = \int_{\mathcal{B}} d^{n-1}\varphi \sqrt{\sigma} T_{ab} n^a \zeta^b, \quad (10)$$

provided the surface  $\mathcal{B}$  contains the orbits of  $\varsigma$ . These quantities are, respectively, the conserved mass, angular and linear momenta of the system enclosed by the boundary  $\mathcal{B}$ . Note that they will both be dependent on the location of the boundary  $\mathcal{B}$  in the spacetime, although each is independent of the particular choice of foliation  $\mathcal{B}$  within the surface  $\partial\mathcal{M}$ .

### III. FOUR-DIMENSIONAL HORIZONLESS DILATON SOLUTIONS

In this section I want to obtain the 4-dimensional horizonless solutions of Eqs. (2)-(4). First, a spacetime generated by a magnetic source which produces a longitudinal magnetic field along the  $z$ -axis is constructed, and second, I obtain a spacetime generated by a magnetic source that produces angular magnetic fields along the  $\varphi$  coordinate.

#### A. The Longitudinal Magnetic Field Solutions

I assume that the metric has the following form:

$$ds^2 = -\frac{\rho^2}{l^2} R^2(\rho) dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\varphi^2 + \frac{\rho^2}{l^2} R^2(\rho) dz^2, \quad (11)$$

where the constant  $l$  have dimension of length which is related to the cosmological constant  $\Lambda$  in the absence of a dilaton field ( $\Phi = 0$ ). Note that the coordinate  $z$  ( $-\infty < z < \infty$ ) has the dimension of length, while the angular coordinate  $\varphi$  is dimensionless as usual and ranges in  $0 \leq \varphi < 2\pi$ . The motivation for this metric gauge [ $g_{tt} \propto -\rho^2$  and  $(g_{\rho\rho})^{-1} \propto g_{\varphi\varphi}$ ] instead of the usual Schwarzschild gauge [ $(g_{\rho\rho})^{-1} \propto g_{tt}$  and  $g_{\varphi\varphi} \propto \rho^2$ ] comes from the fact that I am looking for a magnetic solution instead of an electric one.

The Maxwell equation (2) for the metric (11) is  $\partial_\mu [\rho^2 R^2(\rho) \exp(-2\alpha\Phi) F^{\mu\nu}] = 0$ , which shows that if one choose

$$R(\rho) = \exp(\alpha\Phi), \quad (12)$$

then the vector potential is

$$A_\mu = -\frac{ql}{\rho}\delta_\mu^\varphi, \quad (13)$$

where  $q$  is the charge parameter. The field equations (3) and (4) for the metric (11) can be written as:

$$f'(1 + \alpha\rho\Phi') + \alpha f[\rho\Phi'' + 2\rho\alpha\Phi'^2 + 4\Phi' + (\alpha\rho)^{-1}] - q^2\rho^{-3}e^{-2\alpha\Phi} + \Lambda\rho e^{2\beta\Phi} = 0, \quad (14)$$

$$\frac{\rho}{2}f'' + f'(1 + \alpha\rho\Phi') + 2f[\alpha\rho\Phi'' + \rho(1 + \alpha^2)\Phi'^2 + 2\alpha\Phi'] + \frac{q^2}{\rho^3}e^{-2\alpha\Phi} + \Lambda\rho e^{2\beta\Phi} = 0, \quad (15)$$

$$\frac{\rho}{2}f'' + f'(1 + \alpha\rho\Phi') + q^2\rho^{-3}e^{-2\alpha\Phi} + \Lambda\rho e^{2\beta\Phi} = 0, \quad (16)$$

$$f\Phi'' + f'\Phi' + 2\alpha f\Phi'^2 + 2\rho^{-1}f\Phi' + \alpha q^2\rho^{-4}e^{-2\alpha\Phi} - \beta\Lambda e^{2\beta\Phi} = 0, \quad (17)$$

where “prime” denotes differentiation with respect to  $\rho$ . Subtracting Eq. (15) from Eq. (16) gives:

$$\alpha\rho\Phi'' + 2\alpha\Phi' + \rho(1 + \alpha^2)\Phi'^2 = 0,$$

which shows that  $\Phi(\rho)$  can be written as:

$$\Phi(\rho) = \frac{\alpha}{1 + \alpha^2} \ln\left(\frac{b}{\rho} + c\right), \quad (18)$$

where  $b$  and  $c$  are two arbitrary constants. Using the expression (18) for  $\Phi(\rho)$  in Eqs. (14)-(17), one finds that these equations are inconsistent for  $c \neq 0$ . Thus,  $c$  should vanish.

The only case that I find exact solutions for an arbitrary values of  $\Lambda$  (including  $\Lambda = 0$ ) with  $R(\rho)$  and  $\Phi(\rho)$  of Eqs. (12) and (18) is when  $\beta = \alpha$ . It is easy, then, to obtain the function  $f(\rho)$  as

$$f(\rho) = \rho^{2\gamma} \left( \frac{\Lambda V_0(1 + \alpha^2)^2}{\alpha^2 - 3} \rho^{2(1-2\gamma)} + \frac{m}{\rho} - \frac{(1 + \alpha^2)q^2}{V_0\rho^2} \right), \quad (19)$$

where  $\gamma = \alpha^2/(1 + \alpha^2)$  and  $V_0 = b^{2\gamma}$ . In the absence of a non-trivial dilaton ( $\alpha = 0 = \gamma$ ), the solution reduces to the asymptotically AdS horizonless magnetic string for  $\Lambda = -3/l^2$  [6]. As one can see from Eq. (19), there is no solution for  $\alpha = \sqrt{3}$  with a Liouville potential ( $\Lambda \neq 0$ ).

In order to study the general structure of this solution, one may first look for curvature singularities. It is easy to show that the Kretschmann scalar  $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$  diverges at  $\rho = 0$  and therefore one might think that there is a curvature singularity located at  $\rho = 0$ . However, as will be seen below, the spacetime will never achieve  $\rho = 0$ . Second, one looks for the

existence of horizons, and therefore one searches for possible black hole solutions. The horizons, if any exist, are given by the zeros of the function  $f(\rho) = g^{\rho\rho}$ . Let us denote the smallest positive root of  $f(\rho) = 0$  by  $r_+$ . The function  $f(\rho)$  is negative for  $\rho < r_+$ , and therefore one may think that the hypersurface of constant time and  $\rho = r_+$  is the horizon. However, this analysis is not correct. Indeed, one may note that  $g_{\rho\rho}$  and  $g_{\phi\phi}$  are related by  $f(\rho) = g_{\rho\rho}^{-1} = l^{-2}g_{\phi\phi}$ , and therefore when  $g_{\rho\rho}$  becomes negative (which occurs for  $\rho < r_+$ ) so does  $g_{\phi\phi}$ . This leads to an apparent change of signature of the metric from  $+2$  to  $-2$ , and therefore indicates that an incorrect extension is used. To get rid of this incorrect extension, one may introduce the new radial coordinate  $r$  as

$$r^2 = \rho^2 - r_+^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_+^2} dr^2. \quad (20)$$

With this new coordinate, the metric (11) is

$$ds^2 = -\frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} dt^2 + l^2 f(r) d\phi^2 + \frac{r^2}{(r^2 + r_+^2) f(r)} dr^2 + \frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} dz^2, \quad (21)$$

where  $f(r)$  is now given as

$$f(r) = (r^2 + r_+^2)^\gamma \left( \frac{\Lambda V_0 (1 + \alpha^2)^2}{(\alpha^2 - 3)} (r^2 + r_+^2)^{(1-2\gamma)} + \frac{m}{(r^2 + r_+^2)^{1/2}} - \frac{(1 + \alpha^2) q^2}{V_0 (r^2 + r_+^2)} \right). \quad (22)$$

The gauge potential in the new coordinate is

$$A_\mu = \frac{ql}{(r^2 + r_+^2)^{1/2}} \delta_\mu^\varphi. \quad (23)$$

Now it is a matter of calculation to show that the Kretschmann scalar does not diverge in the range  $0 \leq r < \infty$ . However, the spacetime has a conic geometry and has a conical singularity at  $r = 0$ , since:

$$\lim_{r \rightarrow 0} \frac{1}{r} \sqrt{\frac{g_{\varphi\varphi}}{g_{rr}}} = \frac{1}{2} m l r_+^{2(\gamma-1)} + \frac{2(1 + \alpha^2)}{(\alpha^2 - 3)} \Lambda l V_0 r_+^{1-2\gamma} \neq 1.$$

That is, as the radius  $r$  tends to zero, the limit of the ratio “circumference/radius” is not  $2\pi$  and therefore the spacetime has a conical singularity at  $r = 0$ . In order to investigate the casual structure of the spacetime, I consider it for different ranges of  $\alpha$  separately.

For  $\alpha > \sqrt{3}$ , as  $r$  goes to infinity the dominant term in Eq. (22) is the second term, and therefore the function  $f(r)$  is positive in the whole spacetime, despite the sign of the

cosmological constant  $\Lambda$ , and is zero at  $r = 0$ . Thus, the solution given by Eqs. (21) and (22) exhibits a spacetime with conic singularity at  $r = 0$ .

For  $\alpha < \sqrt{3}$ , the dominant term for large values of  $r$  is the first term, and therefore the function  $f(r)$  given in Eq. (22) is positive in the whole spacetime only for negative values of  $\Lambda$ . In this case the solution presents a spacetime with conic singularity at  $r = 0$ . The solution is not acceptable for  $\alpha < \sqrt{3}$  with positive values of  $\Lambda$ , since the function  $f(r)$  is negative for large values of  $r$ .

Of course, one may ask for the completeness of the spacetime with  $r \geq 0$  [26]. It is easy to see that the spacetime described by Eq. (21) is both null and timelike geodesically complete for  $r \geq 0$ . To do this, one may show that every null or timelike geodesic starting from an arbitrary point either can be extended to infinite values of the affine parameter along the geodesic or will end on a singularity at  $r = 0$ . Using the geodesic equation, one obtains

$$\dot{t} = \frac{l^2}{V_0(r^2 + r_+^2)^{1-\gamma}}E, \quad \dot{z} = \frac{l^2}{V_0(r^2 + r_+^2)^{1-\gamma}}P, \quad \dot{\phi} = \frac{1}{l^2 f(r)}L, \quad (24)$$

$$r^2 \dot{r}^2 = (r^2 + r_+^2)f(r) \left[ \frac{l^2(E^2 - P^2)}{V_0(r^2 + r_+^2)^{1-\gamma}} - \kappa \right] - \frac{r^2 + r_+^2}{l^2}L^2, \quad (25)$$

where the overdot denotes the derivative with respect to an affine parameter, and  $\kappa$  is zero for null geodesics and +1 for timelike geodesics.  $E$ ,  $L$ , and  $P$  are the conserved quantities associated with the coordinates  $t$ ,  $\phi$ , and  $z$ , respectively. Notice that  $f(r)$  is always positive for  $r > 0$  and zero for  $r = 0$ .

First, I consider the null geodesics ( $\kappa = 0$ ). (i) If  $E > P$  the spiraling particles ( $L > 0$ ) coming from infinity have a turning point at  $r_{tp} > 0$ , while the nonspiraling particles ( $L = 0$ ) have a turning point at  $r_{tp} = 0$ . (ii) If  $E = P$  and  $L = 0$ , whatever the value of  $r$ ,  $\dot{r}$  and  $\dot{\phi}$  vanish and therefore the null particles moves on the  $z$ -axis. (iii) For  $E = P$  and  $L \neq 0$ , and also for  $E < P$  and any values of  $L$ , there is no possible null geodesic.

Second, I analyze the timelike geodesics ( $\kappa = +1$ ). Timelike geodesics are possible only if  $l^2(E^2 - P^2) > V_0 r_+^{2(1-\gamma)}$ . In this case the turning points for the nonspiraling particles ( $L = 0$ ) are  $r_{tp}^1 = 0$  and  $r_{tp}^2$  given as

$$r_{tp}^2 = \sqrt{[V_0^{-1}l^2(E^2 - P^2)]^{1/(1-\gamma)} - r_+^2}, \quad (26)$$

while the spiraling ( $L \neq 0$ ) timelike particles are bound between  $r_{tp}^a$  and  $r_{tp}^b$  given by  $0 < r_{tp}^a \leq r_{tp}^b < r_{tp}^2$ . Thus, I confirmed that the spacetime described by Eq. (21) is both null and timelike geodesically complete.



## B. The Rotating Longitudinal Magnetic Field Solutions

Now, I want to endow the spacetime solution (21) with a global rotation. In order to add angular momentum to the spacetime, one may perform the following rotation boost in the  $t - \phi$  plane

$$t \mapsto \Xi t - a\varphi, \quad \varphi \mapsto \Xi\varphi - \frac{a}{l^2}t, \quad (27)$$

where  $a$  is a rotation parameter and  $\Xi = \sqrt{1 + a^2/l^2}$ . Substituting Eq. (27) into Eq. (21) one obtains

$$\begin{aligned} ds^2 = & -\frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} (\Xi dt - a d\varphi)^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} \\ & + l^2 f(r) \left( \frac{a}{l^2} dt - \Xi d\varphi \right)^2 + \frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} dz^2, \end{aligned} \quad (28)$$

where  $f(r)$  is the same as  $f(r)$  given in Eq. (22). The gauge potential is now given by

$$A_\mu = -\frac{q}{(r^2 + r_+^2)^{1/2}} \left( \frac{a}{l} \delta_\mu^t - \Xi l \delta_\mu^\varphi \right). \quad (29)$$

The transformation (27) generates a new metric, because it is not a permitted global coordinate transformation [27]. This transformation can be done locally but not globally. Therefore, the metrics (21) and (28) can be locally mapped into each other but not globally, and so they are distinct. Note that this spacetime has no horizon and curvature singularity. However, it has a conical singularity at  $r = 0$ . It is notable to mention that this solution reduces to the solution of Einstein-Maxwell equation introduced in [6] as  $\alpha$  goes to zero.

The mass and angular momentum per unit length of the string when the boundary  $\mathcal{B}$  goes to infinity can be calculated through the use of Eqs. (8) and (9),

$$\mathcal{M} = \frac{V_0[(3 - \alpha^2)\Xi^2 - 2]}{8l(1 + \alpha^2)} m, \quad \mathcal{J} = \frac{(3 - \alpha^2)V_0}{8l(1 + \alpha^2)} \Xi m a. \quad (30)$$

For  $a = 0$  ( $\Xi = 1$ ), the angular momentum per unit length vanishes, and therefore  $a$  is the rotational parameter of the spacetime. Of course, one may note that these conserved charges reduce to the conserved charges of the rotating black string obtained in Ref. [8] as  $\alpha \rightarrow 0$ .

## C. The Angular Magnetic Field Solutions

In subsection III A, I found a spacetime generated by a magnetic source which produces a longitudinal magnetic field along the  $z$ -axis. Now, I want to obtain a spacetime generated by

a magnetic source that produces angular magnetic fields along the  $\varphi$  coordinate. Following the steps of subsection III A but now with the roles of  $\varphi$  and  $z$  interchanged, one can directly write the metric and vector potential satisfying the field equations (14)-(17) as

$$ds^2 = -\frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} dt^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + (r^2 + r_+^2) e^{2\alpha\Phi} d\varphi^2 + f(r) dz^2, \quad (31)$$

where  $f(r)$  is given in Eq. (22). The angular coordinate  $\varphi$  ranges in  $0 \leq \varphi < 2\pi$ . The gauge potential is now given by

$$A_\mu = \frac{q}{(r^2 + r_+^2)^{1/2}} \delta_\mu^z. \quad (32)$$

The Kretschmann scalar does not diverge for any  $r$  and therefore there is no curvature singularity. The spacetime (31) is also free of conic singularity.

To add linear momentum to the spacetime, one may perform the boost transformation  $[t \mapsto \Xi t - (v/l)z, z \mapsto \Xi z - (v/l)t]$  in the  $t - z$  plane and obtain

$$ds^2 = -\frac{r^2 + r_+^2}{l^2} e^{2\alpha\Phi} \left( \Xi dt - \frac{v}{l} dz \right)^2 + f(r) \left( \frac{v}{l} dt - \Xi dz \right)^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + (r^2 + r_+^2) e^{2\alpha\Phi} d\varphi^2, \quad (33)$$

where  $v$  is a boost parameter and  $\Xi = \sqrt{1 + v^2/l^2}$ . The gauge potential is given by

$$A_\mu = -\frac{q}{(r^2 + r_+^2)^{1/2}} \left( \frac{v}{l} \delta_\mu^t - \Xi \delta_\mu^z \right). \quad (34)$$

Contrary to transformation (27), this boost transformation is permitted globally since  $z$  is not an angular coordinate. Thus the boosted solution (33) is not a new solution. However, it generates an electric field.

The conserved quantities of the spacetime (33) are the mass and linear momentum along the  $z$ -axis given as

$$\mathcal{M} = \frac{V_0[(3 - \alpha^2)\Xi^2 - 2]}{8l(1 + \alpha^2)} m, \quad \mathcal{P} = \frac{(3 - \alpha^2)V_0}{8l(1 + \alpha^2)} \Xi m v.$$

Now, I calculate the electric charge of the solutions (28) and (33) obtained in this section. To determine the electric field one should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces for the spacetimes with a longitudinal magnetic field is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N},$$

and the electric field is  $E^\mu = g^{\mu\rho} \exp(-2\alpha\Phi) F_{\rho\nu} u^\nu$ . Then the electric charge per unit length  $\mathcal{Q}$  can be found by calculating the flux of the electric field at infinity, yielding

$$\mathcal{Q} = \frac{(\Xi^2 - 1)q}{2l}. \quad (35)$$

Note that the electric charge of the string vanishes, when the string has no angular or linear momenta.

#### IV. THE ROTATING SOLUTIONS IN VARIOUS DIMENSIONS

In this section I look for the uncharged rotating solutions of field equations (2)-(4) in  $n + 1$  dimensions. I first obtain the uncharged static solution and then generalize it to the case of rotating solution with all the rotation parameters.

##### A. Static Solutions

I assume the metric has the following form

$$ds^2 = -\frac{\rho^2}{l^2} e^{2\beta\Phi} dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\phi^2 + \frac{\rho^2}{l^2} e^{2\beta\Phi} d\Omega^2, \quad (36)$$

where  $d\Omega^2$  is the metric of the  $(n - 1)$ -dimensional hypersurface which has zero curvature. The field equations (2)-(4) in the absence of electromagnetic field become

$$f'(1 + \beta\rho\Phi') + \beta f[\rho\Phi'' + (n - 1)(2\Phi' + \beta\rho\Phi'^2) + (n - 2)(\beta\rho)^{-1}] + \frac{2\Lambda\rho}{n - 1} e^{2\beta\Phi} = 0, \quad (37)$$

$$\begin{aligned} \frac{\rho}{n - 1} f'' + f'(1 + \beta\rho\Phi') + \frac{4\Lambda\rho}{(n - 1)^2} e^{2\beta\Phi} \\ + 2(n - 2)\rho f \left[ \beta\rho\Phi'' + 2\beta\Phi' + \rho \left( \beta^2 + \frac{4}{(n - 1)^2} \right) \Phi'^2 \right] = 0, \end{aligned} \quad (38)$$

$$\frac{\rho}{n - 1} f'' + f'(1 + \beta\rho\Phi') + \frac{4\Lambda\rho}{(n - 1)^2} e^{2\beta\Phi} = 0, \quad (39)$$

$$f\Phi'' + f'\Phi' + (n - 1) \left[ \beta f\Phi'^2 + \rho^{-1} f\Phi' - \frac{1}{2} \beta \Lambda e^{2\beta\Phi} \right] = 0. \quad (40)$$

Subtracting Eq. (38) from Eq. (39) gives

$$\beta\rho\Phi'' + 2\beta\Phi' + \rho \left( \beta^2 + \frac{4}{(n - 1)^2} \right) \Phi'^2 = 0,$$

which shows that  $\Phi(\rho)$  can be written as:

$$\Phi(\rho) = \frac{(n - 1)^2 \beta}{4 + (n - 1)^2 \beta^2} \ln \left( \frac{c}{\rho} + d \right), \quad (41)$$

where  $c$  and  $d$  are two arbitrary constants. Substituting  $\Phi(\rho)$  of Eq. (41) into the field equations (37)-(40), one finds that they are consistent only for  $d = 0$ . Putting  $d = 0$ , then  $f(\rho)$  can be written as

$$f(\rho) = \frac{8\Lambda V_0}{(n-1)^3\beta^2 - 4n(n-1)}\rho^{2\Gamma} + m\rho^{1-(n-1)\Gamma}, \quad (42)$$

where

$$V_0 = \Gamma^{-2}c^{2(1-\Gamma)}, \quad \Gamma = 4\{(n-1)^2\beta^2 + 4\}^{-1}. \quad (43)$$

One may note that there is no solution for  $(n-1)^2\beta^2 - 4n = 0$ . For the two cases of  $(n-1)^2\beta^2 - 4n > 0$  with positive  $\Lambda$ , and  $(n-1)^2\beta^2 - 4n < 0$  with negative  $\Lambda$ , the function  $f(\rho)$  is positive in the whole spacetime. In these two cases, since the Kretschmann scalar diverges at  $\rho = 0$  and  $f(\rho)$  is positive for  $\rho > 0$ , the spacetimes present naked singularities. For  $(n-1)^2\beta^2 - 4n < 0$  with positive  $\Lambda$ , the function  $f(\rho)$  is negative for  $\rho > r_c$ , where  $r_c$  is the root of  $f(\rho) = 0$ . In this case, the signature of the metric (36) will be changed as  $\rho$  becomes larger than  $r_c$ , and therefore this case is not acceptable.

The only case that there exist horizonless solutions with conic singularity is when  $(n-1)^2\beta^2 - 4n > 0$  with negative  $\Lambda$ . In this case, as  $r$  goes to infinity the dominant term is the second term of Eq. (42), and therefore  $f(\rho) > 0$  for  $\rho > r_+$ , where  $r_+$  is the root of  $f(\rho) = 0$ . Again, one should perform the transformation (20). Thus, the horizonless solution can be written as

$$ds^2 = -\frac{r^2 + r_+^2}{l^2}e^{2\beta\Phi}dt^2 + l^2 f(r)d\phi^2 + \frac{r^2}{(r^2 + r_+^2)f(r)}dr^2 + \frac{r^2 + r_+^2}{l^2}e^{2\beta\Phi}dz^2, \quad (44)$$

$$f(r) = \frac{8\Lambda V_0}{(n-1)^3\beta^2 - 4n(n-1)}(r^2 + r_+^2)^\Gamma + m(r^2 + r_+^2)^{\frac{1}{2}[1-(n-1)\Gamma]}, \quad (45)$$

where  $\Gamma$  and  $V_0$  are given in Eq. (43). Again this spacetime has no horizon and no curvature singularity. However, it has a conical singularity at  $r = 0$ . One should note that this solution reduces to the  $(n+1)$ -dimensional uncharged solution of Einstein gravity given in [7] for  $\beta = 0$  ( $\Gamma = 1 = V_0$ ).

## B. Rotating solutions with all the rotation parameters

The rotation group in  $(n+1)$ -dimensions is  $SO(n)$  and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators,

which is  $[n/2] \equiv k$ , where  $[n/2]$  is the integer part of  $n/2$ . The generalization of the metric (44) with all rotation parameters is

$$\begin{aligned}
ds^2 = & -\frac{r^2 + r_+^2}{l^2} e^{2\beta\Phi} \left( \Xi dt - \sum_{i=1}^k a_i d\phi^i \right)^2 + f(r) \left( \sqrt{\Xi^2 - 1} dt - \frac{\Xi}{\sqrt{\Xi^2 - 1}} \sum_{i=1}^k a_i d\phi^i \right)^2 \\
& + \frac{r^2 dr^2}{(r^2 + r_+^2) f(r)} + \frac{r^2 + r_+^2}{l^2 (\Xi^2 - 1)} e^{2\beta\Phi} \sum_{i < j}^k (a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 + r_+^2}{l^2} e^{2\beta\Phi} dX^2, \\
\Xi^2 = & 1 + \sum_{i=1}^k \frac{a_i^2}{l^2},
\end{aligned} \tag{46}$$

where  $a_i$ 's are  $k$  rotation parameters,  $f(r)$  is given in Eq. (45), and  $dX^2$  is now the Euclidean metric on the  $(n - k - 1)$ -dimensional submanifold with volume  $V_{n-k-1}$ .

The conserved mass and angular momentum per unit volume  $V_{n-k-1}$  of the solution calculated on the boundary  $\mathcal{B}$  at infinity can be calculated through the use of Eqs. (8) and (9),

$$\mathcal{M} = (2\pi)^k \frac{\Gamma^n V_0^{(n-1)/2}}{16\pi l^{n-k-1}} \left\{ \left[ n - \frac{(n-1)^2 \beta^2}{4} \right] \Xi^2 - (n-1) \right\} m, \tag{47}$$

$$\mathcal{J}_i = (2\pi)^k \frac{\Gamma^n V_0^{(n-1)/2}}{16\pi l^{n-k-1}} \left\{ n - \frac{(n-1) \beta^2 \Xi^2}{4} \right\} \Xi m a_i. \tag{48}$$

Note that for Einstein gravity the above computed conserved quantities reduce to those given in [7].

## V. CLOSING REMARKS

Till now, no explicit rotating charged dilaton solutions have been found except for some dilaton coupling such as  $\alpha = \sqrt{3}$  [18] or  $\alpha = 1$  when the string three-form  $H_{abc}$  is included [28]. For general dilaton coupling, the properties of charged dilaton black holes have been investigated only for rotating solutions with infinitesimally small angular momentum [19] or small charge [20]. It has also been shown numerically that for the case of rotating solutions  $\alpha = \sqrt{3}$  is a critical value while for larger values of  $\alpha$  new effect can appear [21]. In this paper I obtained two classes of exact horizonless rotating solutions with Liouville-type potentials in four dimensions, provided  $\beta = \alpha \neq \sqrt{3}$ . These solutions are neither asymptotically flat nor (A)dS. The first class of solutions yields a 4-dimensional spacetime with a longitudinal magnetic field [the only nonzero component of the vector potential is  $A_\varphi(r)$ ] generated by

a static magnetic string. I also found the rotating spacetime with a longitudinal magnetic field by a rotational boost transformation. These solutions have no curvature singularity and no horizons, but have conic singularity at  $r = 0$ . In these spacetimes, when the rotation parameter is zero (static case), the electric field vanishes, and therefore the string has no net electric charge. For the spinning string, when the rotation parameter is nonzero, the string has a net electric charge density which is proportional to the rotation parameter  $a$ . The second class of solutions yields a spacetime with angular magnetic field. These solutions have no curvature singularity, no horizon, and no conic singularity. Again, I found that the string in these spacetimes have no net electric charge when the boost parameter vanishes. I also showed that, for the case of the traveling string with nonzero boost parameter, the net electric charge density of the string is proportional to the magnitude of its velocity ( $v$ ). These solutions reduce to the magnetic rotating string of [6] as  $\alpha \rightarrow 0$ . I also computed the conserved quantities of the four-dimensional magnetic string through the use of the counterterm method.

Next, I generalized the rotating uncharged solutions to arbitrary  $n+1$  dimensions with all rotation parameters. These spacetimes present naked singularities for  $[(n-1)^2\beta^2 - 4n < 0, \Lambda < 0]$  and  $[(n-1)^2\beta^2 - 4n > 0, \Lambda > 0]$ . The only case that the spacetime exhibits a horizonless dilaton string with conic singularity is when  $(n-1)^2\beta^2 - 4n > 0$  and  $\Lambda < 0$ .

Note that the  $(n+1)$ -dimensional rotating solutions obtained here are uncharged. Thus, it would be interesting if one can construct rotating solutions in  $(n+1)$  dimensions in the presence of dilaton and electromagnetic fields. One may also attempt to generalize these kinds of solutions obtained here to the case of two-term Liouville potential [15]. The case of charged rotating black string in four dimensions and  $(n+1)$ -dimensional rotating black branes will be present elsewhere.

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